Fuzzy equivalence relations and their equivalence classes

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Abstract: Fuzzy sets were introduced in 1965 by L. A. Zadeh, as a method for representing some imprecise aspects of human knowledge that would be used in dealing with problems when the source of imprecision is the absence of sharply defined criteria of class membership. Such problems are very often when one deals with classes of objects encountered in the real physical world, and for that reason fuzzy sets have significant applications in many scientific fields. The concept of a fuzzy relation naturally arose from that of fuzzy set in Zadeh's very first paper on fuzzy sets, and it was further developed in his 1971 paper [12], where the notions of a fuzzy equivalence relation and fuzzy ordering were introduced. After that, a number of papers dealing with various aspects related with these relations have appeared, and today, the theory of fuzzy binary relations is probably one of the most important and influential branches of fuzzy set theory. By allowing intermediate degrees of relationship, fuzzy relations provide much more freedom to express the subtle nuances of human preferences so they found natural applications in modeling various concepts inherent to so-called 'soft' sciences like psychology, sociology, linguistics, etc.

Fuzzy equivalence relations generalize the crisp equivalence relations and equality to the fuzzy framework. They have been widely studied as a way to measure the degree of indistinguishability or similarity between the objects of a given universe of discourse, and they have shown to be useful in different contexts such as fuzzy control, approximate reasoning, cluster analysis, etc. Depending on the authors and the context in which they appears, they have

received other names such as similarity relations (original Zadeh's name) or indistinguishability operators (used by Valverde, Boixader, Jacas, Recasens and others).

The key point in work with fuzzy equivalence relations is definition of transitivity (i.e. of composition of fuzzy relations). The most widely used approach, proposed by Zadeh, is based on transitivity defined using MIN operation on the real unit interval [0,1], and in more recent development transitivity is defined in a more general way, by means of arbitrary triangular norms and conorms or using other structures of truth values, such as complete residuated lattices, Heyting algebras etc. In this paper we work with all of these ways for defining transitivity. First of all, we consider various questions concerning properties of fuzzy equivalence classes, close to the ones treated by Nemitz [9], Murali [8], Ovchinnikov [10], Kuroki [7], and others. In particular, we prove that the set of all fuzzy equivalence relations on a set A having a given fuzzy subset f of A as its equivalence class is the closed interval of the lattice $\mathcal{E}(A)$ of all fuzzy equivalence relation on A. We also give necessary and sufficient conditions for a family of fuzzy subsets of A to be a set of equivalence classes of some fuzzy equivalence relation on A, and give a new description of fuzzy partitions.

It turned out that the considered questions are closely related to the ones appearing in study of T-indistinguishability operators initiated by Valverde [11], and further developed in a series of papers by Jacas, Recasens, Boixader, Demirci and others [1]–[5]. From that reason we also study various questions proposed by them, such as generating of a fuzzy equivalence relation by a given family of fuzzy sets, construction of minimal generating family of a given fuzzy equivalence relation, questions concerning extensionality (observability) with respect to a fuzzy equivalence, etc.

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